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A DIFFERENTIAL MODEL OF PLASTICALLY DEFORMED VISCOELASTIC MATERIAL

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In the present paper a model of a plastically deformed viscoelastic material is constructed that takes into account the following features of the material's behavior: 1) elastic strains reach large values; 2) the free energy of the material cannot be expressed in explicit form in terms of the first, second, and third invariants of the Finger or Cauchy–Green strain measure; and 3) after removal of external loads and completion of relaxation processes, the material must enter an unloaded state, no matter what inhomogeneous stress field exists in the medium.

Such a behavior is typical of unstitched polymers in a highly elastic state. The free energy of the polymers is often expressed in terms of a function of principal elongations, which is difficult to represent by an analytical dependence on strain invariants [1–3].

The relaxation processes in these materials have been described by differential (e.g., [4–6]) and integral [7–9] models. We describe the relaxation processes using internal parameters. Taking account of the latter in the expression of free energy by means of a quadratic form, one can construct a reasonable mathematical model. The degree of the medium's elastic and plastic deformation is quantitatively determined by the corresponding tensors. Various methods of decomposition of the measure of complete strain into elastic and plastic components [10–17] are known. We use the Lie decomposition [16], which is determined with accuracy to plastic rotation [18]. It is suggested to determine the tensor characteristic of the plastic-strain rate by analyzing the rates of changes in the principal elastic elongations. The equation of plastic flow is formulated without using the objective derivative. All expressions and equalities are written in coordinates that are convenient for solving problems by the modified Lagrange method [19].

1. Notation. Let us use lower-case letters for scalar quantities, semiboldface small letter for vector characteristics, and upper-case letters for second-order tensor quantities.

The symbols t_0, t, and t_* designate the initial, current, and reference (chosen arbitrarily) times; $t_0 \leqslant t_* < t$. The term "reference" is used for time t_*, because all determining equations are formulated in the $\mathbf{r}_*$ coordinates of the points of the medium at time t_*. The radius vectors of these points at the initial t_0 and current t times are denoted by $\mathbf{r}_0$ and $\mathbf{r}$, and the position of the points in an unloaded state of equilibrium is denoted by vector $\mathbf{r}_\alpha$. In other words, the position of continuum points is described by the radius vector $\mathbf{r}_\alpha$ if the material in a current state is instantly unloaded, and all transient processes are completed.

Below the following tensors are used:

$$\mathbf{Q}_\alpha = \frac{\partial \mathbf{r}(t, \mathbf{r}_\alpha)}{\partial \mathbf{r}_\alpha}, \quad \mathbf{Q}_A = \frac{\partial \mathbf{r}_\alpha(t, \mathbf{r}_*)}{\partial \mathbf{r}_*}, \quad \mathbf{Q}_R = \frac{\partial \mathbf{r}(t, \mathbf{r}_*)}{\partial \mathbf{r}_*}.$$

We shall need a unit tensor $\mathbf{E}$ and the position-gradient operators

$$\nabla \ldots = \mathbf{e}_i \frac{\partial}{\partial x^i} \ldots, \quad \nabla_* \ldots = \mathbf{e}_i \frac{\partial}{\partial x_*^i} \ldots,$$

where $\mathbf{r} = x^i \mathbf{e}_i$; $\mathbf{r}_* = x_*^i \mathbf{e}_i$; and $\mathbf{e}_i$ are the basis vectors of the Cartesian coordinate system. Below, in mathematical expressions, the subscript in angle brackets near the closing parenthesis $(\ldots)_{\langle i \rangle}$ denotes the

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absence of summation over it; the symbol I_3^* is the third invariant of the tensor $\mathbf{Q}_R^t \cdot \mathbf{Q}_R$:

$$I_3^* = I_3(\mathbf{Q}_R^t \cdot \mathbf{Q}_R).$$

2. Derivation of the Relation Between the Rate of Principal Elastic Elongations and the Geometric Tensor Characteristics of Medium. We consider the Cauchy–Green elastic-strain measure

$$\mathbf{G} = \left(\frac{\partial \mathbf{r}}{\partial \mathbf{r}_\alpha}\right)^t \cdot \left(\frac{\partial \mathbf{r}}{\partial \mathbf{r}_\alpha}\right) = \mathbf{Q}_\alpha^t \cdot \mathbf{Q}_\alpha. \tag{2.1}$$

Let the normalized vectors $\mathbf{g}_i$ be its eigenvectors, and the squares of quantities λ_i the eigenvalues. The rotation tensor $\mathbf{O}$ relates the three orthonormalized basis vectors of the Cartesian coordinate system $\mathbf{e}_i$ to the three orthonormalized vectors $\mathbf{g}_i$:

$$\mathbf{g}_i = \mathbf{O} \cdot \mathbf{e}_i. \tag{2.2}$$

The Cauchy–Green elastic-strain measure can be written as

$$\mathbf{G} = \lambda_i^2 \mathbf{g}_i \mathbf{g}_i = \lambda_i^2 \mathbf{O} \cdot \mathbf{e}_i \mathbf{e}_i \cdot \mathbf{O}^t, \tag{2.3}$$

and the squares of elastic elongations along the principal axes λ_i^2 can be determined from the formulas

$$\lambda_i^2 = (\mathbf{e}_i \mathbf{e}_i)_{\langle i \rangle} \cdot\cdot \mathbf{O}^t \cdot \mathbf{G} \cdot \mathbf{O}. \tag{2.4}$$

Let us consider the time variation of the characteristic λ_i. We differentiate equality (2.4) with respect to time t. Using the skew-symmetric tensor

$$\mathbf{W} = \frac{\partial \mathbf{O}^t}{\partial t} \cdot \mathbf{O} = -\mathbf{W}^t$$

we write it as

$$\begin{aligned}\left(2\lambda_i \frac{\partial \lambda_i}{\partial t}\right)_{\langle i \rangle} &= (\mathbf{e}_i \mathbf{e}_i)_{\langle i \rangle} \cdot\cdot (\mathbf{W} \cdot \mathbf{O}^t \cdot \mathbf{G} \cdot \mathbf{O} + \mathbf{O}^t \cdot \frac{\partial \mathbf{G}}{\partial t} \cdot \mathbf{O} + \mathbf{O}^t \cdot \mathbf{G} \cdot \mathbf{O} \cdot \mathbf{W}^t) \\ &= (\mathbf{e}_i \mathbf{e}_i)_{\langle i \rangle} \cdot\cdot (\mathbf{W} \cdot \lambda_j^2 \mathbf{e}_j \mathbf{e}_j + \mathbf{O}^t \cdot \frac{\partial \mathbf{G}}{\partial t} \cdot \mathbf{O} + \lambda_j^2 \mathbf{e}_j \mathbf{e}_j \cdot \mathbf{W}^t).\end{aligned}$$

The rules of tensor convolution and the skew-symmetry of tensor $\mathbf{W}$ allow us to simplify this expression:

$$\left(2\lambda_i \frac{\partial \lambda_i}{\partial t}\right)_{\langle i \rangle} = (\mathbf{e}_i \mathbf{e}_i)_{\langle i \rangle} \cdot\cdot \mathbf{O}^t \cdot \frac{\partial \mathbf{G}}{\partial t} \cdot \mathbf{O}.$$

Thus, the problem of determining the law of λ_i variations with time reduces to analysis of the dependence of the derivative on tensor $\mathbf{G}$.

Note once more that our aim is to formulate all equations in the t and $\mathbf{r}_*$ coordinates. This also concerns the form of tensor $\mathbf{G}$. This can be realized via the relation

$$\mathbf{Q}_\alpha = \frac{\partial \mathbf{r}}{\partial \mathbf{r}_\alpha} = \frac{\partial \mathbf{r}}{\partial \mathbf{r}_*} \cdot \frac{\partial \mathbf{r}_*}{\partial \mathbf{r}_\alpha} = \frac{\partial \mathbf{r}}{\partial \mathbf{r}_*} \cdot \left(\frac{\partial \mathbf{r}_\alpha}{\partial \mathbf{r}_*}\right)^{-1} = \mathbf{Q}_R \cdot \mathbf{Q}_A^{-1}. \tag{2.5}$$

As a result, the Cauchy–Green elastic-strain measure is given by

$$\mathbf{G} = (\mathbf{Q}_A^t)^{-1} \cdot \mathbf{Q}_R^t \cdot \mathbf{Q}_R \cdot \mathbf{Q}_A^{-1}. \tag{2.6}$$

One can take a derivative of this expression with respect to time t. Using the formula of inverse tensor differentiation

$$\frac{\partial \mathbf{Q}_A^{-1}}{\partial t} = -\mathbf{Q}_A^{-1} \cdot \frac{\partial \mathbf{Q}_A}{\partial t} \cdot \mathbf{Q}_A^{-1},$$

we write the derivative

$$\frac{\partial \mathbf{G}}{\partial t} = -(\mathbf{Q}_A^t)^{-1} \cdot \frac{\partial \mathbf{Q}_A^t}{\partial t} \cdot (\mathbf{Q}_A^t)^{-1} \cdot \mathbf{Q}_R^t \cdot \mathbf{Q}_R \cdot \mathbf{Q}_A^{-1} + (\mathbf{Q}_A^t)^{-1} \cdot \frac{\partial \mathbf{Q}_R^t}{\partial t} \cdot \mathbf{Q}_R \cdot \mathbf{Q}_A^{-1}$$
$$+ (\mathbf{Q}_A^t)^{-1} \cdot \mathbf{Q}_R^t \cdot \frac{\partial \mathbf{Q}_R}{\partial t} \cdot \mathbf{Q}_A^{-1} - (\mathbf{Q}_A^t)^{-1} \cdot \mathbf{Q}_R^t \cdot \mathbf{Q}_R \cdot \mathbf{Q}_A^{-1} \cdot \frac{\partial \mathbf{Q}_A}{\partial t} \cdot \mathbf{Q}_A^{-1}. \tag{2.7}$$

We change the form of the right-hand side of Eq. (2.7) by including the convolution with a unit tensor into the second and third terms

$$\frac{\partial \mathbf{G}}{\partial t} = -(\mathbf{Q}_A^t)^{-1} \cdot \frac{\partial \mathbf{Q}_A^t}{\partial t} \cdot (\mathbf{Q}_A^t)^{-1} \cdot \mathbf{Q}_R^t \cdot \mathbf{Q}_R \cdot \mathbf{Q}_A^{-1} + (\mathbf{Q}_A^t)^{-1} \cdot (\mathbf{Q}_R^t \cdot (\mathbf{Q}_R^t)^{-1}) \cdot \frac{\partial \mathbf{Q}_R^t}{\partial t} \cdot \mathbf{Q}_R \cdot \mathbf{Q}_A^{-1}$$
$$+ (\mathbf{Q}_A^t)^{-1} \cdot \mathbf{Q}_R^t \cdot \frac{\partial \mathbf{Q}_R}{\partial t} \cdot (\mathbf{Q}_R^{-1} \cdot \mathbf{Q}_R) \cdot \mathbf{Q}_A^{-1} - (\mathbf{Q}_A^t)^{-1} \cdot \mathbf{Q}_R^t \cdot \mathbf{Q}_R \cdot \mathbf{Q}_A^{-1} \cdot \frac{\partial \mathbf{Q}_A}{\partial t} \cdot \mathbf{Q}_A^{-1}.$$

Relations (2.5) and (2.6) allow this expression to be written in a shorter form:

$$\frac{\partial \mathbf{G}}{\partial t} = -(\mathbf{Q}_A^t)^{-1} \cdot \frac{\partial \mathbf{Q}_A^t}{\partial t} \cdot \mathbf{G} + \mathbf{Q}_\alpha^t \cdot (\mathbf{Q}_R^t)^{-1} \cdot \frac{\partial \mathbf{Q}_R^t}{\partial t} \cdot \mathbf{Q}_\alpha + \mathbf{Q}_\alpha^t \cdot \frac{\partial \mathbf{Q}_R}{\partial t} \cdot \mathbf{Q}_R^{-1} \cdot \mathbf{Q}_\alpha - \mathbf{G} \cdot \frac{\partial \mathbf{Q}_A}{\partial t} \cdot \mathbf{Q}_A^{-1}. \tag{2.8}$$

For further analysis we need to define concretely the meaning of tensor $\mathbf{Q}_\alpha$. According to the theorem of polar decomposition of tensors, the equality $\mathbf{Q}_\alpha = \mathbf{O}_\alpha \cdot \mathbf{V}$ with positive symmetric tensor $\mathbf{V}$ and orthogonal tensor $\mathbf{O}_\alpha$ is valid. From representations (2.1) and (2.3), the direct form of tensor $\mathbf{V}$ can be determined:

$$\mathbf{V} = \lambda_i \mathbf{O} \cdot \mathbf{e}_i \mathbf{e}_i \cdot \mathbf{O}^t.$$

Therefore tensor $\mathbf{Q}_\alpha$ is written as

$$\mathbf{Q}_\alpha = \mathbf{O}_f \cdot \lambda_i \mathbf{e}_i \mathbf{e}_i \cdot \mathbf{O}^t, \tag{2.9}$$

where the rotation tensor is

$$\mathbf{O}_f = \mathbf{O}_\alpha \cdot \mathbf{O}. \tag{2.10}$$

It is easy to verify that the same orthogonal tensor results from the representation of the eigenvectors $\mathbf{h}_i$ of the Finger elastic-strain measure $\mathbf{F}$ in terms of the basis vectors $\mathbf{e}_i$ of the Cartesian coordinate system. Note that the eigenvalues of tensor $\mathbf{F}$ will be the quantities λ_i^2, which, at the same time, are the eigenvalues of tensor $\mathbf{G}$:

$$\mathbf{F} = \left(\frac{\partial \mathbf{r}}{\partial \mathbf{r}_\alpha}\right) \cdot \left(\frac{\partial \mathbf{r}}{\partial \mathbf{r}_\alpha}\right)^t = \lambda_i^2 \mathbf{O}_f \cdot \mathbf{e}_i \mathbf{e}_i \cdot \mathbf{O}_f^t;$$
$$\mathbf{h}_i = \mathbf{O}_f \cdot \mathbf{e}_i. \tag{2.11}$$

We substitute the values of tensors $\mathbf{Q}_\alpha$ and $\mathbf{G}$ (2.3) and (2.9) into Eq. (2.8):

$$\frac{\partial \mathbf{G}}{\partial t} = -(\mathbf{Q}_A^t)^{-1} \cdot \frac{\partial \mathbf{Q}_A^t}{\partial t} \cdot \lambda_i^2 \mathbf{O} \cdot \mathbf{e}_i \mathbf{e}_i \cdot \mathbf{O}^t + \mathbf{O} \cdot \lambda_i \mathbf{e}_i \mathbf{e}_i \cdot \mathbf{O}_f^t \cdot (\mathbf{Q}_R^t)^{-1} \cdot \frac{\partial \mathbf{Q}_R^t}{\partial t} \cdot \mathbf{O}_f \cdot \lambda_j \mathbf{e}_j \mathbf{e}_j \cdot \mathbf{O}^t$$
$$+ \mathbf{O} \cdot \lambda_i \mathbf{e}_i \mathbf{e}_i \cdot \mathbf{O}_f^t \cdot \frac{\partial \mathbf{Q}_R}{\partial t} \cdot \mathbf{Q}_R^{-1} \cdot \mathbf{O}_f \cdot \lambda_j \mathbf{e}_j \mathbf{e}_j \cdot \mathbf{O}^t - \lambda_i^2 \mathbf{O} \cdot \mathbf{e}_i \mathbf{e}_i \cdot \mathbf{O}^t \cdot \frac{\partial \mathbf{Q}_A}{\partial t} \cdot \mathbf{Q}_A^{-1}.$$

Then, the derivative of the parameter λ_i with respect to time (2.5) can be rewritten as

$$\frac{\partial \lambda_i}{\partial t} = -\left(\frac{1}{2\lambda_i} \mathbf{e}_i \mathbf{e}_i\right)_{\langle i \rangle} \cdot\cdot \mathbf{O}^t \cdot \left((\mathbf{Q}_A^t)^{-1} \cdot \frac{\partial \mathbf{Q}_A^t}{\partial t} \cdot \lambda_j^2 \mathbf{O} \cdot \mathbf{e}_j \mathbf{e}_j\right)$$
$$+ \left(\frac{1}{2\lambda_i} \mathbf{e}_i \mathbf{e}_i\right)_{\langle i \rangle} \cdot\cdot \left(\lambda_j \mathbf{e}_j \mathbf{e}_j \cdot \mathbf{O}_f^t \cdot (\mathbf{Q}_R^t)^{-1} \cdot \frac{\partial \mathbf{Q}_R^t}{\partial t} \cdot \mathbf{O}_f \cdot \lambda_k \mathbf{e}_k \mathbf{e}_k\right)$$
$$+ \left(\frac{1}{2\lambda_i} \mathbf{e}_i \mathbf{e}_i\right)_{\langle i \rangle} \cdot\cdot \left(\lambda_j \mathbf{e}_j \mathbf{e}_j \cdot \mathbf{O}_f^t \cdot \frac{\partial \mathbf{Q}_R}{\partial t} \cdot \mathbf{Q}_R^{-1} \cdot \mathbf{O}_f \cdot \lambda_k \mathbf{e}_k \mathbf{e}_k\right)$$

$$-\left(\frac{1}{2\lambda_i}\,\mathbf{e}_i\mathbf{e}_i\right)_{\langle i\rangle}\cdot\cdot\left(\lambda_j^2\mathbf{e}_j\mathbf{e}_j\cdot\mathbf{O}^{\mathrm{t}}\cdot\frac{\partial\mathbf{Q}_A}{\partial t}\cdot\mathbf{Q}_A^{-1}\right)\cdot\mathbf{O}$$

$$=-\left(\frac{1}{2}\,\lambda_i\mathbf{e}_i\mathbf{e}_i\right)_{\langle i\rangle}\cdot\cdot\mathbf{O}^{\mathrm{t}}\cdot\left((\mathbf{Q}_A^{\mathrm{t}})^{-1}\cdot\frac{\partial\mathbf{Q}_A^{\mathrm{t}}}{\partial t}+\frac{\partial\mathbf{Q}_A}{\partial t}\cdot\mathbf{Q}_A^{-1}\right)\cdot\mathbf{O}$$

$$+\left(\frac{1}{2}\,\lambda_i\mathbf{e}_i\mathbf{e}_i\right)_{\langle i\rangle}\cdot\cdot\mathbf{O}_f^{\mathrm{t}}\cdot\left((\mathbf{Q}_R^{\mathrm{t}})^{-1}\cdot\frac{\partial\mathbf{Q}_R^{\mathrm{t}}}{\partial t}+\frac{\partial\mathbf{Q}_R}{\partial t}\cdot\mathbf{Q}_R^{-1}\right)\cdot\mathbf{O}_f.$$

Using relations (2.2), (2.10), and (2.11), we obtain the final required expression:

$$\frac{\partial\lambda_i}{\partial t}=(\lambda_i\mathbf{e}_i\mathbf{e}_i)_{\langle i\rangle}\cdot\cdot(\mathbf{O}_f^{\mathrm{t}}\cdot\mathbf{D}_R\cdot\mathbf{O}_f-\mathbf{O}^{\mathrm{t}}\cdot\mathbf{D}_A\cdot\mathbf{O})=(\lambda_i\mathbf{h}_i\mathbf{h}_i)_{\langle i\rangle}\cdot\cdot(\mathbf{D}_R-\mathbf{O}_\alpha\cdot\mathbf{D}_A\cdot\mathbf{O}_\alpha^{\mathrm{t}})$$

$$=(\lambda_i\mathbf{g}_i\mathbf{g}_i)_{\langle i\rangle}\cdot\cdot(\mathbf{O}_\alpha^{\mathrm{t}}\cdot\mathbf{D}_R\cdot\mathbf{O}_\alpha-\mathbf{D}_A)=(\lambda_i\mathbf{h}_i\mathbf{h}_i)_{\langle i\rangle}\cdot\cdot\mathbf{D}_R-(\lambda_i\mathbf{g}_i\mathbf{g}_i)_{\langle i\rangle}\cdot\cdot\mathbf{D}_A. \tag{2.12}$$

Here

$$\mathbf{D}_R=\frac{1}{2}\left((\mathbf{Q}_R^{\mathrm{t}})^{-1}\cdot\frac{\partial\mathbf{Q}_R^{\mathrm{t}}}{\partial t}+\frac{\partial\mathbf{Q}_R}{\partial t}\cdot\mathbf{Q}_R^{-1}\right);\qquad \mathbf{D}_A=\frac{1}{2}\left((\mathbf{Q}_A^{\mathrm{t}})^{-1}\cdot\frac{\partial\mathbf{Q}_A^{\mathrm{t}}}{\partial t}+\frac{\partial\mathbf{Q}_A}{\partial t}\cdot\mathbf{Q}_A^{-1}\right).$$

It is readily seen that tensors $\mathbf{D}_R$ and $\mathbf{D}_A$ are indifferent to motion of the medium as an absolutely rigid body.

3. Basic Thermodynamic Relations. Equations describing the processes occurring in materials must satisfy the first and second laws of thermodynamics. Moreover, their formulation must be invariant to representation in any inertial system. This condition leads to important conclusions [20, 21]. The requirement of invariance to the choice of an inertial system of reference is fulfilled only with satisfaction of the equation of continuity

$$\frac{\partial}{\partial t}(\sqrt{I_3^*}\rho)=0, \tag{3.1}$$

the law of motion of the medium

$$\rho\frac{\partial^2\mathbf{r}}{\partial t^2}-(\mathbf{Q}_R^{\mathrm{t}})^{-1}\cdot\cdot\overset{*}{\nabla}\mathbf{T}=0, \tag{3.2}$$

the law of energy conservation

$$\rho\frac{\partial e}{\partial t}-\mathbf{T}\cdot\cdot\mathbf{D}_R-\mathbf{T}\cdot\cdot\mathbf{W}_D+(\mathbf{Q}_R^{\mathrm{t}})^{-1}\cdot\cdot\overset{*}{\nabla}\mathbf{q}=0, \tag{3.3}$$

and the thermodynamic inequality

$$\rho\frac{\partial f}{\partial t}+\rho s\frac{\partial\theta}{\partial t}-\mathbf{T}\cdot\cdot\mathbf{D}_R-\mathbf{T}\cdot\cdot\mathbf{W}_D+\frac{1}{\theta}\,\overset{*}{\nabla}\theta\cdot\mathbf{Q}_R^{-1}\cdot\mathbf{q}\leqslant 0, \tag{3.4}$$

where ρ is the material mass in a small element of the medium referred to the volume of this element at time t; θ is the temperature; e, f and s are the mass densities of the internal energy, free energy, and entropy of the continuum; $\mathbf{T}$ is the tensor of true stresses in the material (the Cauchy stress tensor); $\mathbf{q}$ is the heat-flux vector; and

$$\mathbf{W}_D=\frac{1}{2}\left(\frac{\partial\mathbf{Q}_R}{\partial t}\cdot\mathbf{Q}_R^{-1}-(\mathbf{Q}_R^{\mathrm{t}})^{-1}\cdot\frac{\partial\mathbf{Q}_R^{\mathrm{t}}}{\partial t}\right).$$

Laws (3.1)–(3.4) were formulated in the $\mathbf{r}_*$ coordinates (in the reference configuration in Lagrangian coordinates). The free energy is related to the internal energy and entropy by the equality

$$f=e-\theta s. \tag{3.5}$$

We consider materials whose free-energy density f is a function of the medium's temperature θ, the characteristics of its reversible strains λ_i, and the parameters ξ_1, ξ_2, and ξ_3 that characterize the features of relaxation processes. It is assumed that

$$f = f_e(\theta, \lambda_1, \lambda_2, \lambda_3) + 0.5c(\xi_1^2 + \xi_2^2 + \xi_3^2), \quad c = c(\theta) \geqslant 0.$$

Here $c = c(\theta)$ is a nonnegative function of the temperature θ; and ξ_1, ξ_2, and ξ_3 are dimensionless quantities.

In our case, the behavior of the medium depends on the following: 1) the coordinates of the deformed element x^1, x^2, and x^3 at current time; 2) the coordinates of the deformed element x_α^1, x_α^2, and x_α^3 from which the elastic strains are counted off (these vary with propagation of plastic flow); 3) the relaxation characteristics ξ_1, ξ_2, and ξ_3 of the current state; and 4) the temperature. The first six quantities and the temperature have a clear physical meaning. The parameters ξ_1, ξ_2, and ξ_3 are introduced to construct a physically valid and simple (from a mathematical viewpoint) model.

4. Equation Describing the Processes in the Material. Let us consider thermodynamic inequality (3.4). The derivative of the free energy with respect to time has the form

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial f}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial t} + \frac{\partial f}{\partial \xi_i}\frac{\partial \xi_i}{\partial t} = \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial f}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial t} + c\xi_i\frac{\partial \xi_i}{\partial t}.$$

Using the notation

$$\sigma_e^i = \rho\lambda_i \frac{\partial f_e}{\partial \lambda_i}; \tag{4.1}$$

$$\sigma_d^i = \frac{\rho c \xi_i}{\eta}, \quad \eta > 0 \tag{4.2}$$

and Eq. (2.12), we rewrite thermodynamic inequality (3.4) as

$$\rho\Big(\frac{\partial f}{\partial \theta} + s\Big)\frac{\partial \theta}{\partial t} + \sigma_e^i \mathbf{h}_i\mathbf{h}_i \cdot\cdot (\mathbf{D}_R - \mathbf{O}_\alpha \cdot \mathbf{D}_A \cdot \mathbf{O}_\alpha^t) + \eta\sigma_d^i \frac{\partial \xi_i}{\partial t} - \mathbf{T}\cdot\cdot\mathbf{D}_R - \mathbf{T}\cdot\cdot\mathbf{W}_D + \frac{1}{\theta}\overset{*}{\nabla}\theta \cdot \mathbf{Q}_R^{-1} \cdot \mathbf{q} \leqslant 0. \tag{4.3}$$

Using the rotation tensor $\mathbf{O}_t$, we can express any three orthonormalized vectors $\mathbf{j}_i$ in terms of the basis vectors $\mathbf{e}_i$ of the Cartesian coordinate system as $\mathbf{j}_i = \mathbf{O}_t \cdot \mathbf{e}_i$, where the time derivative of the rotation tensor $\mathbf{O}_t$ is the convolution of this tensor with the skew-symmetric $\mathbf{W}_t$:

$$\frac{\partial \mathbf{O}_t}{\partial t} = \mathbf{W}_t \cdot \mathbf{O}_t.$$

This implies validity of the identity

$$(\mathbf{j}_i\mathbf{j}_i)_{\langle i\rangle}\cdot\cdot\frac{\partial}{\partial t}(\mathbf{j}_k\mathbf{j}_k)_{\langle k\rangle} = (\mathbf{j}_i\mathbf{j}_i)_{\langle i\rangle}\cdot\cdot\frac{\partial}{\partial t}(\mathbf{O}_t \cdot \mathbf{e}_k\mathbf{e}_k \cdot \mathbf{O}_t^t)_{\langle k\rangle} = (\mathbf{j}_i\mathbf{j}_i)_{\langle i\rangle}\cdot\cdot(\mathbf{W}_t\mathbf{j}_k\mathbf{j}_k + \mathbf{j}_k\mathbf{j}_k \cdot \mathbf{W}_t^t)_{\langle k\rangle} = 0$$

and fulfillment of the relation

$$\sigma_d^i \frac{\partial \xi_i}{\partial t} = \sigma_d^i \mathbf{j}_i\mathbf{j}_i\cdot\cdot\frac{\partial}{\partial t}(\xi_k\mathbf{j}_k\mathbf{j}_k) = \sigma_d^i\mathbf{j}_i\mathbf{j}_i\cdot\cdot\frac{D}{Dt}(\xi_k\mathbf{j}_k\mathbf{j}_k), \tag{4.4}$$

where $D/Dt\ldots$ is an objective derivative of the second-order tensor that satisfies the conditions

$$\mathbf{A}\cdot\cdot\frac{D\mathbf{A}}{Dt} = \mathbf{A}\cdot\cdot\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{E}\cdot\cdot\frac{D\mathbf{A}}{Dt} = \frac{\partial}{\partial t}(\mathbf{E}\cdot\cdot\mathbf{A}) \tag{4.5}$$

for any symmetric tensor $\mathbf{A}$. Using (4.4), we transform expression (4.3):

$$\begin{aligned} &\rho\Big(\frac{\partial f}{\partial \theta} + s\Big)\frac{\partial \theta}{\partial t} + (\sigma_e^i\mathbf{h}_i\mathbf{h}_i + \sigma_d^i\mathbf{j}_i\mathbf{j}_i)\cdot\cdot(\mathbf{D}_R - \mathbf{O}_\alpha \cdot \mathbf{D}_A \cdot \mathbf{O}_\alpha^t) \\ &+ \sigma_d^i\mathbf{j}_i\mathbf{j}_i\cdot\cdot\Big(\eta\frac{D}{Dt}(\xi_k\mathbf{j}_k\mathbf{j}_k) - (\mathbf{D}_R - \mathbf{O}_\alpha \cdot \mathbf{D}_A \cdot \mathbf{O}_\alpha^t)\Big) \\ &- \mathbf{T}\cdot\cdot\mathbf{D}_R - \mathbf{T}\cdot\cdot\mathbf{W}_D + \frac{1}{\theta}\overset{*}{\nabla}\theta \cdot \mathbf{Q}_R^{-1} \cdot \mathbf{q} \leqslant 0. \end{aligned} \tag{4.6}$$

Thus, the requirement of agreement of the mathematical model with the first and second thermodynamic laws reduces to satisfaction of inequality (4.6). There are many methods for constructing the governing equations of material behavior that satisfy this inequality. We consider one of the possible and physically reasonable systems of equations.

Let the continuum behavior be determined by the equations

$$s = -\frac{\partial f}{\partial \theta}; \tag{4.7}$$

$$\mathbf{T} = \mathbf{T}_e + \mathbf{T}_d; \tag{4.8}$$

$$\mathbf{T}_e = \sigma_e^i \mathbf{h}_i \mathbf{h}_i; \tag{4.9}$$

$$\mathbf{T}_d = \sigma_d^i \mathbf{j}_i \mathbf{j}_i; \tag{4.10}$$

$$\mathbf{q} = -\eta_q (\mathbf{Q}_R^t)^{-1} \cdot \overset{*}{\nabla} \theta; \tag{4.11}$$

$$\eta \frac{D\mathbf{U}}{Dt} = -\eta_\xi \mathbf{U} + \Big(\mathbf{D} - \frac{1}{3}(\mathbf{D}\cdot\cdot\mathbf{E})\mathbf{E}\Big), \quad \mathbf{U} = \xi_i \mathbf{j}_i \mathbf{j}_i; \tag{4.12}$$

$$\mathbf{D} = \mathbf{D}_R - \mathbf{O}_\alpha \cdot \mathbf{D}_A \cdot \mathbf{O}_\alpha^t; \tag{4.13}$$

$$\mathbf{O}_\alpha \cdot \mathbf{D}_A \cdot \mathbf{O}_\alpha^t = \nu \eta_A \Big(\mathbf{T} - \frac{1}{3}(\mathbf{T}\cdot\cdot\mathbf{E})\mathbf{E}\Big), \tag{4.14}$$

$$\eta_A \geqslant 0, \quad \eta_\xi \geqslant 0, \quad \eta_q \geqslant 0, \quad 0 \leqslant \nu \leqslant 1.$$

Equality (4.7) expresses the well-known statement of thermodynamics. The mass density of the medium's entropy s equals the derivative of the mass density of the equilibrium free energy f_e with respect to the temperature θ with opposite sign.

Expressions (4.8)–(4.10) and relations (4.1) and (4.2) are known in mechanics. The stresses occurring in the continuum are the sum of the equilibrium $\mathbf{T}_e$ and dissipative $\mathbf{T}_d$ components. The main values of the equilibrium components of stresses σ_e^i are uniquely determined by the derivative of the equilibrium mass density of the medium's free energy f_e with respect to the elastic elongations λ_i, and the relaxation characteristics of the state of the material ξ_i can be described by parameters that are proportional to the main values σ_d^i of dissipative stresses $\mathbf{T}_d$. In this case, the spatial orientation of equilibrium stresses $\mathbf{T}_e$ depends on the eigenvectors $\mathbf{h}_i$ of the Finger strain measure $\mathbf{F}$. The spatial orientation of dissipative stresses $\mathbf{T}_d$ is determined by the eigenvectors $\mathbf{j}_i$ of the relaxation tensor $\mathbf{U}$ in which the parameters ξ_i are the eigenvalues.

Equality (4.11) rewritten with allowance for the relation between the position-gradient operators in actual $\mathbf{r}$ and studied $\mathbf{r}_*$ configurations has the form $\mathbf{q} = -\eta_q \nabla\theta$ and represents the Fourier heat conductivity law.

Relation (4.12) describes relaxation, creep, and viscoelastic processes. It is noteworthy that the parameter η_ξ is a nonnegative function of the state parameters of the medium and their time derivatives. Therefore, in the general case, the relaxation law (4.12) is substantially nonlinear.

Note one more important property. The law (4.12) guarantees the equality of the mean dissipative stresses to zero

$$\frac{1}{3}\mathbf{T}_d\cdot\cdot\mathbf{E} = \frac{1}{3}(\sigma_d^1 + \sigma_d^2 + \sigma_d^3) = 0. \tag{4.15}$$

Prove this. The double convolution of relation (4.10) with a unit tensor $\mathbf{E}$

$$\mathbf{T}_d\cdot\cdot\mathbf{E} = \frac{\rho c \xi_i}{\eta} \mathbf{j}_i \mathbf{j}_i \cdot\cdot\mathbf{E}$$

leads to the conclusion that the equality of the expression $\xi_i \mathbf{j}_i \mathbf{j}_i\cdot\cdot\mathbf{E}$ to zero will suffice for obtaining the necessary result. To determine the values of the expression $\xi_i \mathbf{j}_i \mathbf{j}_i\cdot\cdot\mathbf{E}$, we perform double convolution of the material behavior law (4.12) with a unit tensor. The law studied [with allowance for (4.5)] takes the form

$$\eta \frac{\partial}{\partial t}(\xi_i \mathbf{j}_i \mathbf{j}_i\cdot\cdot\mathbf{E}) = -\eta_\xi (\xi_i \mathbf{j}_i \mathbf{j}_i\cdot\cdot\mathbf{E}).$$

This equation has a trivial solution that satisfies the initial zero data (which characterize the initial unloaded state of the medium):

$$\xi_i \mathbf{j}_i \mathbf{j}_i \cdot\cdot \mathbf{E} = 0. \tag{4.16}$$

Hence, condition (4.15) is valid.

Relation (4.14) is the law of development of plastic deformation and can be rewritten as

$$\mathbf{O}_\alpha \cdot \mathbf{D}_A \cdot \mathbf{O}_\alpha^{\mathrm{T}} = \frac{3 d_{\text{int}}}{2\sigma_{\text{int}}} \Big(\mathbf{T} - \frac{1}{3} (\mathbf{T} \cdot\cdot \mathbf{E}) \mathbf{E} \Big). \tag{4.17}$$

Here

$$\sigma_{\text{int}} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2},$$

where σ_i are the eigenvalues of tensor $\mathbf{T}$, and

$$d_{\text{int}} = \frac{\sqrt{2}}{3} \sqrt{(d_1 - d_2)^2 + (d_2 - d_3)^2 + (d_1 - d_3)^2},$$

where d_i are the eigenvalues of tensor $\mathbf{D}_A$.

We emphasize that law (4.14) can be used when the field of values of the parameter ν at space points is known. In a physical sense, the quantity ν characterizes the degree of development of plastic flow, i.e., it shows to what extent the conditions of compatibility of plastic strains allow plastic flow to develop. In other words, it shows the contribution of actual plastic strain rates developed at a given point of the medium to the maximum rates at a given load (observed on homogeneous samples under similar conditions).

The parameter ν can be calculated from the equality $\nu = (3/2)(1/\eta_A)(d_{\text{int}}/\sigma_{\text{int}})$. However, to this end, it is necessary to determine the field of the intensity distribution of plastic flow velocities d_{int} and the field of stress intensity σ_{int}, using equality (4.17). However, equality (4.17) allows one to evaluate the intensity field of plastic flow velocities d_{int} only with accuracy to a constant cofactor. The latter should be uniquely determined from the requirement that the maximum value of the parameter ν must be equal to unity.

We now must verify the validity of inequality (4.6). It is worth noting that the symmetry of the complete stress tensor $\mathbf{T}$ and the skew-symmetry of the tensor $\mathbf{W}_D$ make the result of their double convolution vanish:

$$\mathbf{T} \cdot\cdot \mathbf{W}_D = 0. \tag{4.18}$$

Using expressions (4.1), (4.12), (4.7)–(4.14), and (4.16), (4.18), we transform inequality (4.6):

$$\begin{aligned} &\mathbf{T} \cdot\cdot (\mathbf{D}_R - \mathbf{O}_\alpha \cdot \mathbf{D}_A \cdot \mathbf{O}_\alpha^{\mathrm{t}}) - \sigma_{\mathrm{d}}^{i} \mathbf{j}_i \mathbf{j}_i \cdot\cdot \Big(\eta_\xi \xi_i \mathbf{j}_i \mathbf{j}_i + \frac{1}{3} (\mathbf{D} \cdot\cdot \mathbf{E}) \mathbf{E} \Big) \\ &\quad - \mathbf{T} \cdot\cdot \mathbf{D}_R - \frac{1}{\theta} \eta_q \overset{*}{\nabla} \theta \cdot \mathbf{Q}_R^{-1} \cdot (\mathbf{Q}_R^{\mathrm{t}})^{-1} \overset{*}{\nabla} \theta \\ &= -\mathbf{T} \cdot\cdot \nu \eta_A \Big(\mathbf{T} - \frac{1}{3} (\mathbf{T} \cdot\cdot \mathbf{E}) \mathbf{E} \Big) - \frac{\eta_\xi c \rho}{\eta} \xi_i \xi_i - \frac{1}{\theta} \eta_q \overset{*}{\nabla} \theta \cdot \mathbf{Q}_R^{-1} \cdot (\mathbf{Q}_R^{\mathrm{t}})^{-1} \cdot \overset{*}{\nabla} \theta \leqslant 0. \end{aligned}$$

We finally write this inequality as

$$\nu \eta_A \Big(\mathbf{T} \cdot\cdot \mathbf{T} - \frac{1}{3} (\mathbf{T} \cdot\cdot \mathbf{E})^2 \Big) + \frac{\eta_\xi c \rho}{\eta} \xi_i \xi_i + \frac{1}{\theta} \eta_q \overset{*}{\nabla} \theta \cdot \mathbf{Q}_R^{-1} \cdot (\mathbf{Q}_R^{\mathrm{t}})^{-1} \cdot \overset{*}{\nabla} \theta \geqslant 0.$$

Obviously, the second and third terms in the expression cannot take negative values. The same is true for the first term and is proved by the transform

$$\mathbf{T} \cdot\cdot \mathbf{T} - \frac{1}{3} (\mathbf{T} \cdot\cdot \mathbf{E})^2 = \sigma_i \sigma_i - \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3)^2 = \frac{1}{3} ((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2) \geqslant 0. \tag{4.19}$$

To model the processes occurring in the medium, it is necessary to formulate a heat conduction equation. To derive it, we substitute, using (3.5), the expression of mass density e via the characteristics

f, θ, and s into the law of energy conservation (3.3):

$$\rho\left(\frac{\partial f}{\partial t}+s\,\frac{\partial \theta}{\partial t}+\theta\,\frac{\partial s}{\partial t}\right)-\mathbf{T}\cdot\cdot\mathbf{D}_R-\mathbf{T}\cdot\cdot\mathbf{W}_D+(\mathbf{Q}_R^{\mathrm{t}})^{-1}\cdot\cdot\overset{*}{\nabla}\mathbf{q}=0. \qquad (4.20)$$

The value of the time derivative of f in equality (4.20) should be determined from the formulas given below. The first of these is obtained from relations (2.12), (4.1), (4.9), and (4.14):

$$\rho\,\frac{\partial f}{\partial \lambda_j}\,\frac{\partial \lambda_j}{\partial t}=(\sigma_e^i\mathbf{h}_i\mathbf{h}_i)\cdot\cdot(\mathbf{D}_R-\mathbf{O}_\alpha\cdot\mathbf{D}_A\cdot\mathbf{O}_\alpha^{\mathrm{t}})=\mathbf{T}_e\cdot\cdot\mathbf{D}_R-\mathbf{T}_e\cdot\cdot\nu\eta_A\Big(\mathbf{T}-\frac{1}{3}\,(\mathbf{T}\cdot\cdot\mathbf{E})\mathbf{E}\Big).$$

The second formula is derived from equalities (4.2), (4.4), (4.10), and (4.12)–(4.15):

$$\rho\,\frac{\partial f}{\partial \xi_k}\,\frac{\partial \xi_k}{\partial t}=(\sigma_d^i\mathbf{j}_i\mathbf{j}_i)\cdot\cdot\eta\,\frac{\partial}{\partial t}\,(\xi_k\mathbf{j}_k\mathbf{j}_k)=-\mathbf{T}_d\cdot\cdot\Big(\eta_\xi(\xi_i\mathbf{j}_i\mathbf{j}_i)-\mathbf{D}_R+\mathbf{O}_\alpha\cdot\mathbf{D}_A\cdot\mathbf{O}_\alpha^{\mathrm{t}}+\frac{1}{3}\,(\mathbf{D}\cdot\cdot\mathbf{E})\mathbf{E}\Big)$$

$$=-\mathbf{T}_d\cdot\cdot\Big(\eta_\xi(\xi_i\mathbf{j}_i\mathbf{j}_i)-\mathbf{D}_R+\nu\eta_A\Big(\mathbf{T}-\frac{1}{3}\,(\mathbf{T}\cdot\cdot\mathbf{E})\mathbf{E}\Big)\Big).$$

Finally, substituting the above formulas and relations (4.7) and (4.18) into expression (4.20), we obtain

$$\rho\theta\,\frac{\partial s}{\partial t}-\mathbf{T}\cdot\cdot\nu\eta_A\Big(\mathbf{T}-\frac{1}{3}\,(\mathbf{T}\cdot\cdot\mathbf{E})\mathbf{E}\Big)-\mathbf{T}_d\cdot\cdot\eta_\xi\xi_k\mathbf{j}_k\mathbf{j}_k+(\mathbf{Q}_R^{\mathrm{t}})^{-1}\cdot\cdot\overset{*}{\nabla}\mathbf{q}=0.$$

Taking into account equalities (4.2), (4.10), and (4.19), we finally formulate Eq. (4.20) as the law

$$\rho\theta\,\frac{\partial s}{\partial t}=\pi_\xi+\pi_A-(\mathbf{Q}_R^{\mathrm{t}})^{-1}\cdot\cdot\overset{*}{\nabla}\mathbf{q},\quad \pi_\xi=\frac{\eta_\xi c\rho}{\eta}\,\xi_i\xi_i,\quad \pi_A=\frac{1}{3}\,\nu\eta_A((\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2), \qquad (4.21)$$

where π_ξ is the heat release due to dissipative losses in relaxation processes, and π_A is the heat release due to the plastic deformation of the material. This relation is the heat conduction equation. The entropy of the system changes because of heat exchange between the medium and neighboring regions, the heat release during relaxation transitions from one medium's state to another, and in the development of plastic deformations.

5. Conclusions. Within the framework of the above model, the processes occurring in the material are described by continuity equations (3.1), by equations of motion (3.2), relaxation of the mechanical properties of the medium (4.12), and development of plastic deformations (4.14), and by heat-conduction equation (4.21).

The material's properties are defined by the scalar functions f, η_A, η_ξ, η_q, and η. These functions must be determined experimentally. In this case, the quantities η_A, η_ξ, η_q, and η can be functions of the state parameters of the material θ, λ_1, λ_2, λ_3, ξ_1, ξ_2, and ξ_3 and of their time derivatives. It is important, however, that the inequalities $\eta_A \geqslant 0$, $\eta_\xi \geqslant 0$, $\eta_q \geqslant 0$, and $\eta > 0$ be satisfied. The parameter ν must obey the constraint $0 \leqslant \nu \leqslant 1$.

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